Solutions to Practice Midterm 2

1. (a) $-\frac{1}{\sqrt{n}} \leq \frac{\sin n}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$. By the Squeeze Theorem, $\lim_{n\to\infty} \frac{\sin n}{\sqrt{n}} = 0$.

$\quad -\frac{1}{\sqrt{n}} \to 0, \quad \frac{1}{\sqrt{n}} \to 0$

(b) Let $f(x) = x\sin\left(\frac{1}{x}\right)$. Then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$

$$\underset{(\text{substitute } t = 1/x)}{=} \lim_{t\to 0} \frac{\sin t}{t} \underset{(\text{l'Hôpital})}{=} \lim_{t\to 0} \frac{\cos t}{1} = 1.$$

2. (20 points). Compute, using the method for surface area of a solid of revolution, the surface area of a sphere of radius R.

Solution We can take the function $f(x) = \sqrt{R^2 - x^2}$ on the domain $[-R, R]$. When rotated this gives a sphere of radius R. The derivative of this function is $-\frac{x}{\sqrt{R^2-x^2}}$. Let S denote the surface area. We have

$$\begin{aligned}
S &= 2\pi \int_{-R}^{R} f(x)\sqrt{1 + f'(x)^2}\, dx \\
&= 2\pi \int_{-R}^{R} \sqrt{1 + \frac{x^2}{\sqrt{R^2 - x^2}^2}}\sqrt{R^2 - x^2}\, dx \\
&= 2\pi \int_{-R}^{R} \sqrt{R^2 - x^2 + x^2}\, dx \\
&= 2\pi \int_{-R}^{R} R\, dx \\
&= 2\pi Rx|_{-R}^{R} \\
&= 4\pi R^2.
\end{aligned}$$

3. (20 points). Compute the indefinite integral

$$\int \frac{x+7}{x^2(x+2)}\, dx.$$

Solution First, we write the integrand as a partial fraction

$$\frac{x+7}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2},$$

where we determine A, B, C now. Clear denominators to obtain

$$x + 7 = Ax(x+2) + B(x+2) + Cx^2 = (A+C)x^2 + (2A+B)x + 2B.$$

This yields the system of equations

$$\begin{aligned} A + C &= 0 \\ 2A + B &= 1 \\ 2B &= 7. \end{aligned}$$

Solving, we see that $B = \frac{7}{2}$, $A = -\frac{5}{4}$, and $C = \frac{5}{4}$. Thus, the integral is

$$\begin{aligned} \int \frac{x+7}{x^2(x+2)}\,dx &= \int \left(-\frac{5}{4x} + \frac{7}{2x^2} + \frac{5}{4(x+2)} \right) dx \\ &= -\frac{5}{4}\int \frac{dx}{x} + \frac{7}{2}\int \frac{dx}{x^2} + \frac{5}{4}\int \frac{dx}{x+2} \\ &= -\frac{5}{4}\ln|x| - \frac{7}{2x} + \frac{5}{4}\ln|x+2| + C. \end{aligned}$$

4. (20 points) Find an interval $[a, b]$ containing 0 such that if x is in $[a, b]$, the error of the $5th$ Taylor polynomial for $f(x) = e^x$ (with $a = 0$) is less than or equal to 10^{-18}.

Solution The error bound is

$$|T_5(x) - e^x| \le \frac{K_6(x-0)^6}{6!},$$

where K_6 is an upper bound for $f^6(x) = e^x$ on some interval as yet to be determined. So we solve for x in the inequality

$$\frac{K_6 x^6}{6!} \le 10^{-18}.$$

Let $b_0 = 1$, so that $e^{b_0} = e$. We have $b_0 > 0$ and on the interval $(-\infty, b_0]$, $e^x \le e$. So, for any x in the interval $(-\infty, b_0]$, the error is at most

$$\frac{ex^6}{6!} = \frac{ex^6}{360}.$$

Since $e \le 3$, $\frac{e}{3} \le 1$. Thus,

$$\frac{ex^6}{360} \le \frac{x^6}{120}.$$

Set

$$x^6 \le 120 \cdot 10^{-18}.$$

We can require even more strictly that

$$x^6 \le 2^6 \cdot 10^{-18} = 64 \cdot 10^{-18}.$$

Then, we see that

$$|x| \leq 2 \cdot 10^{-3}$$

has the indicated error. Thus, on the interval $[-\frac{1}{500}, \frac{1}{500}]$, the error

$$|T_5(x) - e^x|$$

is less than or equal to 10^{-18}.

5. (20 points). Compute the value of $\ln 2$ to an error of at most 10^{-3}. You should use Taylor polynomials, but you do not have to actually simplify the final approximation $T_n(2)$.

Solution We use the Taylor polynomials $T_n(x)$ for $f(x) = \ln x$ centered at $a = 1$. The general form for the error bound is then,

$$|T_n(2) - \ln 2| \leq \frac{K_{n+1}(2-1)^{n+1}}{(n+1)!},$$

where K_{n+1} is an upper bound for $f^{(n+1)}(x)$ on the interval $[1, 2]$. The first few derivatives of $f(x)$ are

$$f'(x) = \frac{1}{x}$$
$$f^{(2)}(x) = -\frac{1}{x^2}$$
$$f^{(3)}(x) = \frac{2}{x^3}$$
$$\cdots$$

The absolute values of these derivatives are all decreasing on the interval $[1, 2]$, so we can take their values at $x = 1$ as upper bounds. In fact, we can take

$$K_{n+1} = n!$$

for $n \geq 0$. Now, we have that $|f^{(n+1)}(x)| \leq K_{n+1}$ on the interval $[1, 2]$. Thus, we can take

$$|T_n(2) - \ln 2| \leq \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

Thus,

$$|T_{999}(2) - \ln 2| \leq 10^{-3}.$$

The corresponding approximation is

$$T_{999}(2) = \Sigma_{k=1}^{999}(-1)^{k-1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{1}{999}.$$